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J. W. L. GLAISHER, M.A., F.R.S, President, in the Chair.

W. Austin Ashe, Director of the Observatory, Quebec; and
Gustavus William Nicolls, Caixa 162, Pernambuco, Brazil,

were balloted for, and duly elected Fellows of the Society.

On the Images formed by Reflecting Mirrors, and their Aberration.

By the Hon. Lord McLaren.

The application of reflecting telescopes to photography renders it desirable that their suitability for large fields should be tested by an accurate investigation of the caustic images produced by the reflection of oblique pencils from their surfaces. This is the purpose of the present paper.

It has been suggested that for large fields a mirror of spherical figure possesses some advantages over that of the parabolic form, on the ground that the spherical mirror, while it does not bring any parallel pencil to an accurate focus, reflects always a circular image, and gives images of equal intensity at all angles. The present investigation shows that for fields of any angle likely to be required in practice—say for any angle under two degrees—the lateral aberration of the paraboloid is inconsiderable. But it also brings out a variation of focal length for different distances from the centre of the field, which even for a field of 2° is considerable. It will be shown in the sequel that the cycloidal mirror possesses some advantages in this respect over mirrors of the parabolic type.

The form of the caustic surface produced by direct reflexion from a spherical surface is well known: it is, near the image, a

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cone generated by the revolution of an epicycloid, and presents its apex, or cusp, to the observer's eye. I have not found, however, in any work accessible to me, a solution of the problem of the caustic surfaces produced by the reflexion of oblique pencils from a paraboloid. Sir William Hamilton, in his great Memoir on the "Theory of Systems of Rays,"* finds expressions for the caustics formed by reflecting surfaces of any form. The distinguished author has not shown how these expressions are to be integrated for parabolic reflecting surfaces, and I am not aware that his differential equations have been integrated for the general equation of a surface of the second degree.

The solution which I offer for the case of a parabolic mirror is obtained in a different way, and is perhaps of some practical interest. It will be seen that for rays of any inclination to the axis of symmetry the caustic is a curve resembling a parabola, confocal with the reflecting surface, and having its axis inclined to that of the reflector at an angle depending on the inclination of the ray. The equation of the caustic curve, referred to its own axis, is

$$\rho^{\frac{1}{3}} \cos \frac{\theta}{3} = a^{\frac{1}{3}},$$

The small arc of the caustic which constitutes the image of a non-central star is a diminished copy, or nearly so, of the arc of the reflector, but turned round so as to lie nearly in the line of sight. The image of the whole field is convex to the observer's eye, and is aplanatic only in this sense, that equal arcs of the heavens are represented by equal arcs in the image of the field.

CASE I.—*To find the equation to the caustic of the parabola for parallel rays in the principal plane.* (See Plate 7.)

Let $S_1S_2S_0$ be an axial section of the parabolic reflecting surface in the plane of reflection, O the focus and origin of polar coordinates.

S_1O, S_2O two consecutive vectors.

S_1P_1, S_2P_2 two consecutive reflected rays, on which let the perpendiculars OP_1, OP_2 be let fall from the origin.

S_0O (or X) is the axis of the parabolic reflecting surface.
S_0P_0 the reflected ray proceeding from the vertex, and making with the axis an angle $= (\phi)$.
OP_0 is the perpendicular on S_0P_0 from O.

Then the curve $P_1P_2P_0$ represents the locus of the intersections of the reflected rays with the perpendiculars drawn from O, and the curve $C_1P_0C_2$ represents the caustic, or locus of the intersections of consecutive reflected rays.

Let the axis of the reflecting surface be the reference line for the polar equation $f(\rho\,\theta)$ of the parabola, and also for the

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* *Trans. Roy. Irish Acad.* Vol. XV.

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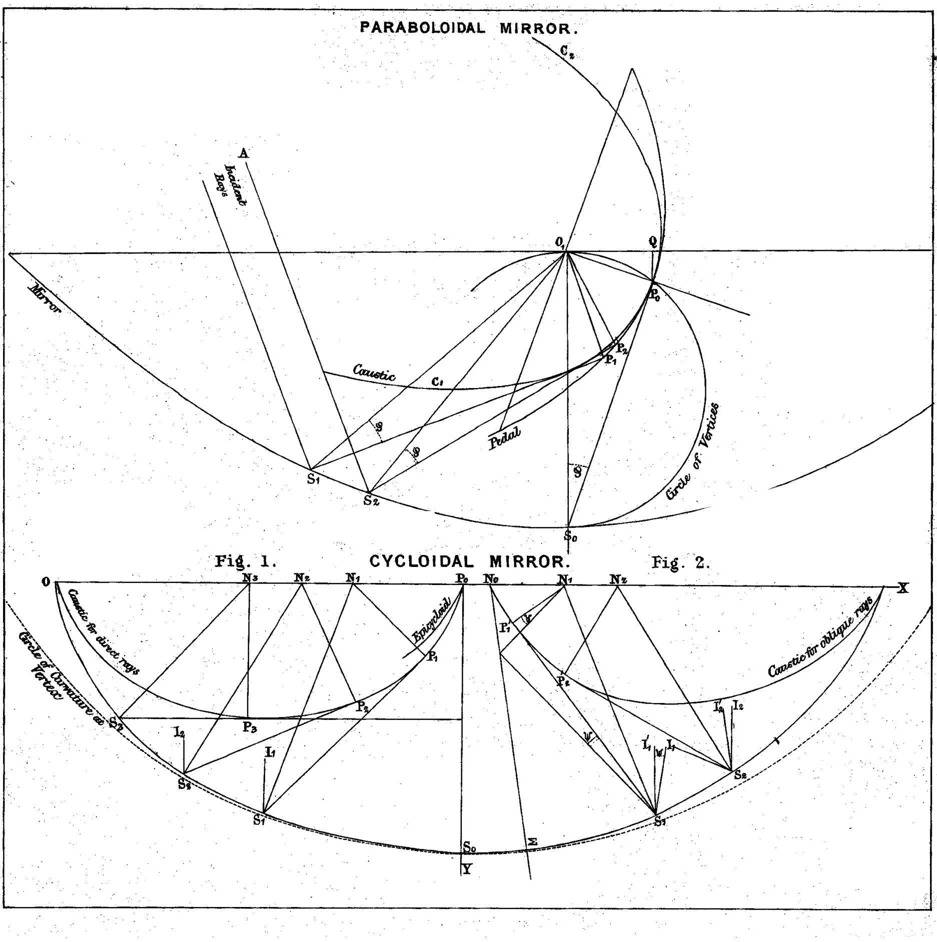

Spottiswoode & Co. Lith. London.

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respective polar equations $f'(\rho' \theta')$ and $f''(\rho'' \theta'')$, of the curve $P_1P_2P_0$ and the curve $C_1P_0C_2$.

Then, because the incident pencil is a pencil of parallel rays, all the rays which are incident on the parabola at points S_1, S_2, &c. are inclined at the same angle $(=\phi)$ to the lines parallel to X which pass through these points; and all the reflected rays are inclined at the same angle $(=-\phi)$ to the corresponding vectors.

Or, if (χ) be the angle between the normal and the x coordinate at any point S_1,
$(-\chi)$ the angle between the normal and the radius vector at the same point,
$(\chi+\phi)$ will be the angle of the incident ray at point S_1,
$-(\chi+\phi)$ will be the angle of the reflected ray at the same point.

In other words, all the reflected rays S_1P_1, S_2P_2, &c. make equal angles $(=-\phi)$ with the vectors passing through S_1, S_2, &c.

Therefore, in the series of triangles OS_1P_1, OS_2P_2, &c. all the angles at S_1, S_2, &c. are $=\phi$;

and all the angles at P_1, P_2, &c. are by construction right angles.

Therefore the remaining angles S_1OP_1, S_2OP_2, &c. are each $=90^\circ-\phi$.

If θ be the angle between the reference line and any vector, OS_1 of the parabola, then

$$\theta'=\theta+(90^\circ-\phi)\ *$$

will be the angle between the reference line and the corresponding vector OP_1 of the curve $P_1P_2P_0$. Also

$$\text{vector } OP_1=\text{vector } OS_1\times\sin\phi,$$

or

$$\rho'=\rho\sin\phi.$$

Now, if the reference line for the curve $P_1P_2P_0$ be turned round to OP_0, while the reference line for the parabola $S_1S_2S_0$ is unchanged, then, for any correlative points S_1P_1 in the two curves whose current coordinates are $(\theta\rho)$ and $(\theta'\rho')$, we have

$$\theta'=\theta,$$

and

$$\rho'=\rho\sin\phi=\rho\times\text{constant}.$$

The curves are therefore similar in figure, and the curve $P_1P_2P_0$ is a parabola with parameter

$$4a'=4a\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where $(4a)$ is the parameter of the reflecting curve.

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* In the figure θ is negative, being in the fourth quadrant.

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The equation of the curve $P_1P_2P_0$ may be thus written:

$$\rho'^{\frac{1}{2}} \cos \tfrac{1}{2}\theta' = a'^{\frac{1}{2}} \quad \ldots\ldots\ldots \quad (b)$$

From this construction the equation of the caustic may be immediately deduced. By construction the derived parabola, $P_1P_2P_0$, is the locus of intersections of perpendiculars drawn from O to the reflected rays. By definition the reflected rays are tangents to the caustic curve, $C_1P_0C_2$, and the caustic is thus the first negative pedal of the derived parabola.

The equation of the caustic is deduced as follows:

By a known property of pedal curves, if

$$\rho'^{m} \cos m\theta' = a^{m}$$

be the equation to a curve,

$$\rho''^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta'' = a^{\frac{m}{m+1}}$$

is the equation of its first negative pedal.*

Hence, from the expression for the derived parabola [Equation (b)]

$$\rho'^{\frac{1}{2}} \cos \tfrac{1}{2}\theta' = a'^{\frac{1}{2}}$$

we may at once write the equation of the caustic.

It is

$$\rho''^{\frac{1}{3}} \cos \tfrac{1}{3}\theta'' = a'^{\frac{1}{3}} \quad \ldots\ldots\ldots \quad (c)$$

where a' is the prime vector in both curves, and is $= a \sin \phi$. The caustic touches the pedal, or derived parabola, at the vertex P_0. For any value of θ'' not greatly exceeding 90° the caustic lies within the pedal parabola.

When $\theta'' = 270°$, $\cos \frac{1}{3}\theta = 0$, and ρ'' becomes infinite.

The caustic is evidently a looped curve, with branches intersecting at the point where $\theta'' = 180°$, and tending to ultimate parallelism with a line at right angles to its axis of symmetry.

Locus of Caustics.—If we suppose the inclination of the pencil to vary we shall have as many caustic curves as there are radiant points on the celestial arc, the quantity a' representing in each case the distance of the star image from the centre of the telescopic field. Before proceeding to the analysis for the surface of revolution, it may be noticed that in the parabolic mirror the locus of the vertices of the caustic curves is the point (P_0), being the angular point of the right-angled triangle OP_0S_0. It therefore lies on a circle (shown in the diagram) whose diameter is the focal length, OS_0, and radius $= \frac{1}{4}$ radius of curvature of the parabola at the vertex. That is to say, all the foci for pencils of different degrees of obliquity (or star-images of variable distances) lie on the circle; and all the

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* Williamson, *Differential Calculus*, p. 231.

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star-images in the field lie on a spherical surface whose radius $= \frac{1}{4}$ radius of curvature of parabola at vertex.

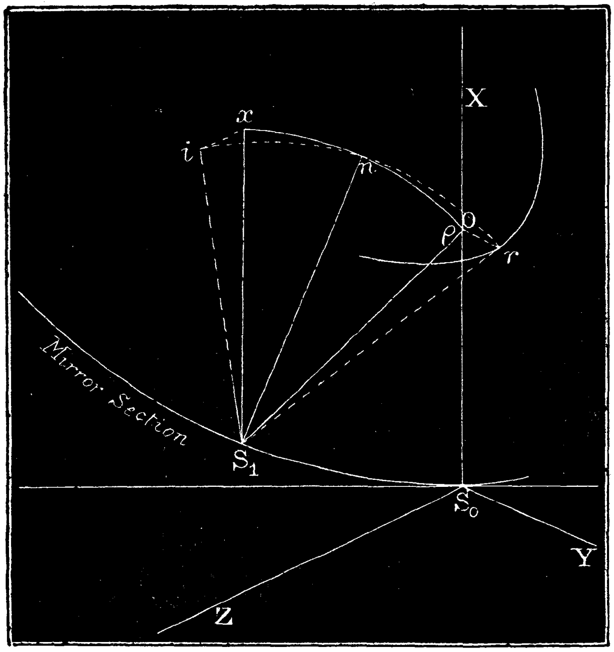

Fig. *a.*

Case II. *Caustic Surface for the Paraboloid of Revolution.*—The caustic surface may be obtained by investigating the equation of the plane curve for any plane section of the paraboloid passing through the axis or at right angles to it.

Except in the case already considered, neither the incident nor the reflected ray lies in the plane of the section, nor do any two consecutive reflected rays lie in the same plane. The solution is therefore a little complicated, but it may be found by projective geometry commencing with the following preliminary proposition.*

The normal, the abscissa of the paraboloid, and the incident ray at any point S constitute a solid angle, whose six edges and angles are respectively equal to those of the solid angle formed by *the normal, the vector, and the reflected ray.* (See Figure *a.*)

If we denote these five lines by their initial letters we have

normal $= n$ abscissa, parallel to $X = x$
incident ray $= i$ radius vector $= \rho$
reflected ray $= r$ S is the incident point.

The same letters, n, x, i, r, and ρ, also denote the points of

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* In what follows the reference plane for projections is a section through the axis of the paraboloid, and having any required inclination to the principal plane. It is assumed that the reflected rays intersect in some regular curve.

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the two spherical triangles from which the angles at S are to be deduced.

To show that the solid angle i, n, x is equal to the solid angle $\rho\, n\, r$.—The three lines x, n, ρ are in one and the same plane, viz. the given axial section of the paraboloid, and angle $n\,\mathrm{S}\,x$=angle $n\,\mathrm{S}\,\rho$. Also the three lines $i\,n\,r$ are in one plane, being the plane of reflexion; and, by the law of reflexion, angle $i\,\mathrm{S}\,n$=angle $n\,\mathrm{S}\,r$. Now these equal angles are proportional to the corresponding sides of the two spherical triangles; also the contained angles formed by the intersection of the plane faces $i\,n\,r\,\mathrm{S}$, $x\,n\,\rho\,\mathrm{S}$ are equal. Therefore the remaining sides and angles are equal.

$$\therefore \angle\, r\mathrm{S}\rho = \angle\, i\mathrm{S}x = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$\angle \text{ between faces } r\mathrm{S}\rho, x\mathrm{S}\rho = \angle \text{ between faces } i\mathrm{S}x, x\mathrm{S}\rho = \psi \quad . \quad . \quad (e)$$

These equations are fundamental. Equation (d) is independent of the particular section. It affirms that for any point S on the surface of a paraboloidal mirror the plane angle ($r\mathrm{S}\rho$) between the reflected ray and the radius vector is *invariable*, and is equal to ϕ, the inclination of pencil to mirror axis.

In Equation (e) it is easily seen that the quantity ψ is the angle between a given axial section and the principal plane. This equation affirms that for any point S on the given mirror section the inclination of the plane faces $r_1\mathrm{S}_1\rho_1$, $r_2\mathrm{S}_2\rho_2$, &c. to the plane of the section is constant, and equal to ψ.

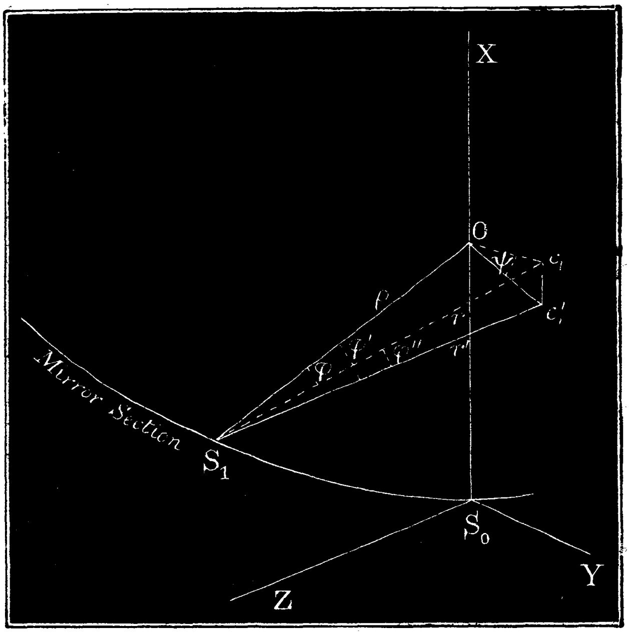

Fig. 6.

From these equations it follows directly that at the variable point S the projection of the reflected ray on the plane of the section makes a constant angle (ϕ') with the radius vector, and

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also that the projection of the reflected ray makes a constant angle with the reflected ray itself. (See Figure *b*, where c_1 is a point in the caustic, and c_1' a point in its projection on the plane of the axial section, OS_1c_1'.)

Because if we consider any of the solid angles formed at S_1, S_2, &c. by the vector, the reflected ray and the projection of the reflected ray (which may be denoted by r'), we see that its edges and angles are proportional to the angles and sides of a right-angled spherical triangle, of which one of the sides$=\phi$ and one of the angles (other than the right angle)$=\psi$. Now, as these quantities are the same for all the solid angles at S_1, S_2, &c. it follows that all the solid angles have their third edges and three angles respectively equal; or generally, for the solid angles at points S_1, S_2, &c. we have

Three edges respectively $=\psi$, $90°$, $90°-\psi$,

and three plane angles ϕ, ϕ', and ϕ'', whose equations are ϕ, $=\angle$ between incident ray and x.

$$\tan\phi'=\tan\phi\cos\psi=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

$$\sin\phi''=\sin\phi\sin\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

To form the equation of the caustic curve in the plane of projection.

With the same diagram and notation as in Case I., the plane of the paper is now the plane of the given axial section, making the angle (ψ) with the principal plane. The lines AS_1, AS_2, &c. are the projections of the parallel incident rays on the section. They are therefore also parallel, and make a constant angle $f(\phi\,\psi)$ with the corresponding *abscissæ*. SP_1, SP_2, &c. are the projections of the reflected rays, to which the perpendiculars OP_1, OP_2, &c. are drawn from the focus.

Then for any one of the points S_1, S_2, &c. we observe that the plane of projection intersects the plane of reflexion in the normal N; and it is evident that equal angles measured from N, in the plane of reflexion, are represented by equal angles in the plane of projection. Therefore the projections of the incident and reflected rays make equal angles with the normal at any point, S_1, S_2, &c. of the series. Then, as before, $\rho_1\theta_1$, $\rho_1'\theta_1'$, $\rho_1''\theta_1''$, are the current coordinates of the original and the two derived curves, whence we deduce the curve in the plane of projection identically as in Case I.

Angles OS_1P_1, OS_2P_2, &c. $=\phi'$ [$=$constant, by Equation (f)].

Angles $S_1P_1O_1$, $S_2P_2O_2$, &c. are by construction right angles. Therefore

Angles S_1OP_1, $S_2OP_2=90-\phi'=$ constant.

Hence $OP=OS\,.\sin\phi'$, or

$$\rho'=\rho\sin\phi' \qquad\qquad a'=a\sin\phi'$$

θ' (measured from its own axis OP_0) $=\theta$ (measured from x).

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Hence the curve $P_0P_1P_2$, or derived parabola $f(\rho'\theta')$, is the locus of intersections of *perpendiculars on the projections of the reflected rays.*

Its parameter $4a' = 4a \sin \phi'$.

And finally the negative pedal of the derived parabola is the projection of the caustic of the given section, or *locus of intersections of the consecutive projections of the reflected rays.*

It is as before

$$f(\rho''\theta'') = \rho''^{\frac{1}{3}} \cos \frac{\theta''}{3} - a'^{\frac{1}{3}} = 0,$$

where, approximately,

$$a' = a \sin \phi \cos \psi.$$

We thus see that for any axial plane of the paraboloid the *projections* of the reflected rays form a caustic curve, having the same equation as the caustic curve in the principal plane.

By projective geometry two curves which are connected by a projective relation are of the same degree and the same class, and differ only in their constant coefficients.*

We therefore conclude that the equation of the caustic is of the same form as the equation of its projection, and its parameter only remains for determination.

Caustic deduced from its Projection.—The reflected ray from the vertex of the mirror is evidently common to all the axial sections. It is therefore invariable in position and magnitude. The prime vector $\grave{a}$ of the caustic deduced in the principal plane is necessarily also common to all the axial sections; because it is the line drawn from the origin to the extremity of the ray reflected from the mirror vertex. That is to say, for all axial sections of the mirror we have caustic curves, of the same form and with parameter invariable in length and position; and, therefore, if the axial section of the mirror is turned round, so as to generate a surface of revolution, the caustic curve also will generate a surface of revolution about its axis of symmetry.

Circular Section of the Paraboloid.—The result just obtained may be found more easily by considering a circular section of the mirror in the plane YZ.

In such a section let ρ_1, ρ_2, &c. be the vectors drawn from consecutive points S_1, S_2, &c. of the circular section to the focus O. They are evidently of equal length, and their integral is a right cone, having its vertex at O. Let r_1, r_2, &c. be consecutive reflected rays drawn from the same series of points S_1, S_2, &c.

Thenby Equation (d) (which is general for the whole reflecting surface) all the angles $\rho_1 S_1 r_1$, $\rho_2 S_2 r_2$, &c. (although lying in different planes) are equal to one another, being equal to the constant ϕ.

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* Cremona, *Projective Geometry*, Chap. II.

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Let perpendiculars OP_1, OP_2 be drawn from O to the reflected rays r_1, r_2, &c. and we have

$$OP_1 = \rho_1 \sin\phi\,;$$

$$OP_2 = \rho_2 \sin\phi = OP_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h)$$

since all the vectors ρ_1, ρ_2, &c. of the circular section are equal.

The locus of P accordingly is a small circle, described on the surface of a sphere whose radius $\rho' = \rho \sin\phi$.

This small circle is the caustic, because the consecutive reflected rays are tangents to it.

If we now consider two opposite points of the mirror section in the plane of the luminous pencil we see that for one of these the point P lies outside the cone of vectors, and for the other it lies inside the cone. The line joining the two points P is a diameter of the caustic circle; and by drawing the diagram in perspective for two parallel circular sections of the mirror it will be seen that the caustic circles are parallel and co-axial. Their integral is therefore a surface of revolution whose equation is that of the caustic curve in the principal plane.

Lateral Aberration.—In the caustic of the parabola we have not to find a circle of least aberration, because at the focus the aberration is *nil*, and we have only to determine for any particular point or image the diameter of a section of the cone of rays through the plane of the field of view. The aberration, whether in a direction radial or transversal to the field, is to be obtained from the equation to a section of the surface, in terms of the focal length (a), the semi-arc of mirror (θ), and the inclination of pencil (ϕ).

In the principal plane the required aberration is the difference of the distances of the *surface of caustic* and its tangent from centre of field, or focus of paraboloid.

The latter

$$= a'' \sec\theta, \text{ where } a'' = a \sin\phi.$$

The former

$$= \rho'' \doteq \frac{a''}{\cos^3(\frac{1}{3}\theta)} = a'' \sec^3\left(\frac{\theta}{3}\right).$$

Therefore

$$\text{aberration} = a \sin\phi \left[\sec\theta - \sec^3\left(\frac{\theta}{3}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (k)$$

θ is determined by the aperture of mirror.

The rays proceeding from that half of the caustic surface which lies on the negative side of its prime vector constitute the right half of the pencil which enters the eye-piece; those proceeding from the positive side of the prime vector constitute the left half of the visual pencil. It is evident that near the centre of the field the images are symmetrical.

Except near the centre of the field there will obviously be a greater dispersion of the rays in the plane XZ than in

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the plane XY of the incident pencil; and the diameter of the visible image will, apparently, be smaller in proportion to the dispersion.* If we assume that the visual pencil, in the direction XZ, is only effective to the extent of one-half of its angular divergence, then we have for this aberration the quantity

$$\psi = \sqrt{\tfrac{1}{2}},$$

and

$$a'' = a \sin\left(\frac{\phi}{\sqrt{2}}\right).$$

Therefore, by Equation (k), we have, in a direction transverse to radius of field,

$$\text{lateral aberration from centre of image} = a \sin\left(\frac{\phi}{\sqrt{2}}\right)\left[\sec\theta - \sec^3\left(\frac{\theta}{3}\right)\right] \quad . \; (l)$$

The breadth of the image will be twice this quantity.

Example of Computation of Lateral Aberration.

Assume focal length = 16 feet = 192 inches = (a) and diameter of mirror = 24 inches.

The semi-arc of mirror will be about 3° 30′ = (θ)

For a star 1° from centre of field, 1° = (ϕ)

$$\log\sin\phi = 8\cdot2419$$

$$\frac{\phi}{\sqrt{2}} = \frac{1^\circ}{1\cdot4142} = 0^\circ\cdot7072 = 43'$$

$$\log\sin\frac{\phi}{\sqrt{2}} = 8\cdot0972$$

$$\sec\theta = \sec 3^\circ\ 30' \qquad = 1\cdot0019$$

$$\left.\begin{array}{r} \sec\tfrac{1}{3}\theta = \sec 1^\circ\ 10' : \log\sec = \cdot00009 \\ \log\sec^3\left(\frac{\theta}{3}\right) = \cdot00027 \end{array}\right\} . \; \sec^3\left(\frac{\theta}{3}\right) = 1\cdot0006$$

$$\sec\theta - \sec^3\left(\frac{\theta}{3}\right) = 0\cdot0013 = f \text{ (secant).}$$

Then

(1) In the principal plane by Equation (k) we have

$$\log 192 \text{ inches} = 2\cdot2833$$

$$\log\sin\phi = 8\cdot2419$$

$$\log f \text{ (secant)} = \log 0\cdot0013 = -3\cdot1139$$

$$\log \text{lateral aberration} = -3\cdot6391$$

$$\text{lateral aberration} = \cdot0044 \text{ inch.}$$

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* Since this paragraph was written the writer has found, by making a large scale perspective drawing of the course of the rays, considered as tangents to the caustic surface, that the converging pencils are apparently truly symmetrical cones, and that their images are therefore *round*. This agrees with the results obtained experimentally by Mr. Howard Grubb (*Monthly Notices* for April, p. 310); and it is evident from the equations that, if the images are circular near the centre of the field, they will be circular through its entire extent, provided the images are received on a concave plate of the proper curvature, as suggested in Mr. Grubb's paper.

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(2) In the transverse plane by Equation (l) we have

log 192	2·2833
$\log \sin \frac{\phi}{\sqrt{2}}$	8·0972
log 0·0013	−3·1139
log aberration =	−3·4944

transverse lateral aberration from centre of image = ·0031 inch

transverse diameter of image = ·0062 inch.

III. *Elliptic and Hyperbolic Reflecting Surfaces.*

Although the writer has not been able as yet to obtain an exact solution of the caustic for parallel rays incident on an elliptic surface, it may be here mentioned that if, in the case of the ellipse, a similar figure to that of Case I. in this paper be drawn with divergent and convergent pencils, making a given angle with the focal pencils, the locus of intersections of perpendiculars on the reflected pencil will be a small ellipse placed obliquely to the primary ellipse, and the caustic will be the first negative pedal of the derived ellipse. The divergent pencil traced backwards will be found to proceed from a radiating surface conjugate with the caustic surface. The caustic curves thus derived do not differ sensibly from the caustic of the parabola when they are represented graphically, and from the manner of their derivation they are obviously curves involving the cubic radical. It would add needlessly to the bulk of this paper to give the demonstration for the case of the elliptic caustic. By drawing the diagram with divergent and convergent pencils, and using the same notation and analysis as in Case I., the reader will have no difficulty in finding that the caustic of the divergent pencils is the first negative pedal of an ellipse whose axis is inclined to the minor axis of the reflecting ellipse at the angle (ϕ) which the divergent pencil makes with the major axis. The same notation and analysis gives for the hyperbola, with converging incident pencils, and diverging reflected pencils, a caustic whose equation is the first negative pedal of a small hyperbola inclined to the reflecting hyperbola as in the case of the ellipse.

The relation found in Case I. between the caustic of the parabola and the equation of the parabola itself is thus shown to be only a limiting case of a more general theorem which includes all curves of the second degree. For if in the case of the ellipse, for example, the major axis be supposed infinite, then the converging pencil (whose rays are supposed to make a constant angle with those of a focal pencil) becomes a pencil of parallel rays having the given inclination to a pencil parallel to the axis of the reflecting surface.

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IV. *Cycloidal Surfaces of Revolution.*—The investigation of the conditions necessary to the production of a perfect image by reflexion has led to the conclusion that these conditions are better fulfilled by the cycloid than by the parabola. This results from two well-known properties of this curve: (1) the base or line on which the generating circle rolls bisects all the radii of curvature; (2) equal linear intervals measured along the base from its centre correspond to equal angles between the radii of curvature.

From the *first* condition it results that, whatever may be the angular extent of the field of view, the geometrical foci of the images lie on the base-line of the cycloid or on the base-plane of its figure of revolution.

From the *second* condition it results that the field of view is a true stereographic projection of an area of the celestial sphere, equal intervals of arc in the sphere being represented by equal straight lines in the field of view.

Caustic of the Cycloid.—Consider first the case of a direct pencil parallel to the axis of symmetry of the mirror (represented in the left quadrant of the plate). The figure represents an axial section, the base being parallel to the axis of (x), and the axis of symmetry parallel to (y).

I_1S_1, I_2S_2, &c. are the incident rays.

N_1S_1, N_2S_2, &c. the normals.

P_1S_1, P_2S_2, &c. the reflected rays.

$f\,(x\ y)$ denotes the equation of the reflecting surface.

$f\,(x'\ y')$ the equation of the caustic.

$I_1S_1N_1$, $I_2S_2N_2$ are the angles which the normals make with lines parallel to the axis of x.

Then in the equation of the reflecting surface

$$I_1S_1N_1 - I_2S_2N_2 = \delta \tan^{-1}\left(\frac{dx}{dy}\right).$$

In the equation of the caustic $I_1S_1P_1$, $I_2S_2P_2$ are the angles which its tangents make with the axis of x. And because $I_1S_1P_1$, $I_2S_2P_2$ are the sums of the angles of incidence and reflexion

$$I_1S_1P_1 - I_2S_2P_2 = 2\,(I_1S_1N_1 - I_2S_2N_2),$$

or

$$\delta \tan^{-1}\left(\frac{dy'}{dx'}\right) = 2\delta \tan^{-1}\frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (z)$$

That is, the curvature of the caustic, at P_1, measured by the deflection of its tangent for unit increment of arc is double the curvature of the reflecting surface at the corresponding point S_1, measured by the deflection of its normal for an equal increment of arc.

If ϕ_1 ϕ_2 be the angles between consecutive normals to the cycloid and the relative incident rays (parallel to y),

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ϕ_1' ϕ_2' the angles between the same normals and the base x

$$\phi_1-\phi_2=\phi_1'-\phi_2'.$$

If θ_1 θ_2 be angles between the incident rays and the reflected rays which are tangents to the caustic

θ_1' θ_2', the angles made by the normals to the caustic with the base

$$\theta_1-\theta_2=\theta_1'-\theta_2'.$$

Therefore, from Equation (z), we have

$$\theta_1'-\theta_2'=2(\phi_1'-\phi_2').$$

We may now directly compare the angles made by the normals of the two curves with the base-line. If we make the base-line the reference-line for angular measure, then at the point O, where the cycloid and the caustic are in contact, $\phi'=0$, $\theta'=0$, and for any other position the normals drawn from S and P to the respective curves make angles with the base-line, having the relation $\theta'=2\phi'$.

If now we join N_1P_1, N_2P_2, &c. without making any assumption as to the value of the angles at P_1, P_2, &c. we have, by the general equation for reflecting surfaces :*

Length of reflected ray $=\frac{1}{2}$ radius of curvature × cosine of angle of reflexion,

or

$$SP=NS\,.\cos NSP.$$

Therefore the angle at P is a right angle. and NP is a normal to the caustic. In other words, *the normals drawn to the respective curves from the extremities of the reflected ray intersect in the base.* Also

$$NP=NS\,.\cos SNP=NS\,.\cos(\theta'-\phi')=NS\cos\phi',$$

which may be written :

$$\rho'=\rho\cos\phi' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

Now consider ON, the intercept on the base-line made by the normals NS and NP at any point.

If $x=\nu$ be the length of ON, measured from O, and a the radius of the generating circle of the cycloid, we have by the "tangential equation" of the cycloid and trochoid

$$\nu=2a\phi \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (b)$$

as an equation between coordinates of the reflecting surface (Williamson, Diff. Calc. p. 339).

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[*] In the Memoir by Sir William Hamilton, before referred to, this relation is deduced from the general differential equation of the reflected ray, and is shown to be true for any reflecting surface, where the ray is contained in the plane of the greatest or least osculating circle to the mirror (*Trans. Roy. Irish Acad.* Vol. XV. p. 97).

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And in the equation of the caustic we have, for the same values of ON or ν,

$$\theta' = 2\phi' \therefore \nu = a\theta' = 2a'\theta'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

The caustic, therefore, satisfies the "tangential equation" for the cycloid and trochoid generated by a circle of radius $a' = 2a$. It must be a cycloid if its normal vanishes at two points on the base-line. At the point O, where the reflecting curve and the caustic are in contact

$$\text{SN or } \rho = 0 : \rho' = 0 \text{ [by Equation } (a)\text{].}$$

At the centre of base-line

$$\rho = 2a, \ \phi' = \frac{\pi}{2}, \ \rho' = \rho \cos \phi' = 0.$$

The caustic meets the base in two cusps, one of them being coincident with the cusp of the reflecting curve, and the other being at the middle of the base-line.

The caustic is therefore a cycloid with radius of generating circle:

$$a' = \frac{a}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

and base

$$2\pi a' = \tfrac{1}{2} \text{ base of reflecting cycloid} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

The whole caustic surface is generated by the revolution of the caustic section about the axis of symmetry.

In connection with the caustic of oblique pencils we shall have occasion to refer to the simultaneous equations (a) and (c):

$$\rho' = \rho \cos \phi' \qquad \nu = 2a'\theta'.$$

Caustic of Oblique Pencils.

With similar notation and construction (see the right quadrant of the plate), and denoting by (ψ) the inclination of the obliquely incident rays to the axis of symmetry of the reflector, and by Σ the point where the oblique pencil is normal to the surface, we have, as before, from the general equation of reflecting sufaces,

$$\text{SP}' = \text{NS} \,.\, \cos \text{NSP}',$$

and from the preceding analysis we have: angle at P, a right angle: NP′, a normal to the caustic, intersecting NS, the normal to the reflecting surface at a point N in the base-line. Now if we imagine a direct ray and an oblique ray, both incident on the reflector at the same point S, so that the construction in figure (1) is superposed on that of figure (2), we see that the normal SN and the point N are common to both systems. We see

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also that the reflected rays SP and SP′ (of direct and oblique reflexion) are inclined to each other at the angle ψ; also that the normals PN and P′N (of the caustics of direct and oblique reflexion) have the same inclination$=\psi$. Hence for any one point in the base-line,

$$\theta'' = \theta' \mp \psi ;$$

and for *any* two points in the base-line, $(x=\nu_1)$ $(x=\nu_2)$

we have, by subtraction,

$$\theta_1'' - \theta_2'' = \theta_1' - \theta_2' ; \ \nu_1' - \nu_2' = \nu_1 - \nu_2 ;$$

that is, for equal differences of base (or progression of the rolling circle) the differences of deflection of the normals in the two caustics are equal.

The caustic of oblique reflexion is therefore a cycloid satisfying an equation of the form

$$\nu = a''\theta''.$$

It consists of two unequal branches meeting in the focus of the oblique pencil, or rather of two separate cycloids on unequal bases, and together occupying the base of the reflecting cycloid.

Observing that the length of base of the caustic of direct reflexion is

$$\nu = 2\pi a',$$

it is easily seen that for the caustic of oblique reflexion the bases of the two branches are

$$2\pi a'' = 2a'(\pi + \psi) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (f)$$

and

$$2\pi a''' = 2a'(\pi - \psi) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (g)$$

by which the generating radii a'' and a''' are determined.

It is supposed that in the case of the oblique pencil the whole caustic surface is formed by revolution with varying parameter about the line ΣN, which is thus an interior tangent to the surface. For direct pencils the caustic surface is obviously a true figure of revolution about S_0P_0.

The result of a careful investigation of the aberration is that at the circles of least aberration or least confusion there is very little difference between the sphere and the cycloid in the matter of lateral aberration, while if the preceding analysis be well founded there would be little or no longitudinal aberration for the images formed in a cycloidal mirror. The epicycloidal caustic of the circle of curvature falls within the cycloidal caustic in the figure.

The limits of this paper, however, do not admit of the further discussion of this subject.

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Results.

The chief interest of the discussion turns on the possibility of obtaining symmetrical images from the caustics of oblique pencils of parallel rays.

1. In every caustic there are two principal points in the curve to be considered. One of these (usually denoted by the letter q) is the focus of the elementary pencil reflected from the central point of the mirror. The other principal point is the focus of that particular element of the pencil which is a normal to the reflecting surface (F).

In astronomical instruments, where the celestial arc viewed in the field is of less angular extent than the arc of the mirror, either of the points described is a possible focus. The question which of the two is the true visible focus of the infinitely distant point is one which must be considered separately for each kind of reflecting surface; because, in general, a visible image will only be formed at or near the point where the curvature of the caustic is a maximum, and the point of maximum curvature can only be determined from the equation of the caustic itself.

2. Other conditions being supposed equal, that form of mirror is to be preferred in which the visible image is formed at the point q; that is to say, at the focus of the element of the converging pencil which comes from the centre of the mirror.

It is proved in the preceding paper that for a paraboloidal mirror this point constitutes the *vertex* of the cubic radical, which is the caustic. At the vertex $\frac{ds}{dp}$ is a maximum; and a greater number of tangent rays will pass through a small finite arc of the caustic at the vertex than will pass through a consecutive arc of equal length. Near this point is the visible image, and it is a remarkable and apparently unique property of the paraboloid that all its visible images are formed *by the rays proceeding from the central part of the mirror.* Moreover the image point is not a cusp, but only a point of maximum intensity in the luminous surface.

It is possible that there is also a secondary focus at the point F where the oblique pencil is normal to the parabola; but this focus has not been investigated.

3. It has been shown for the case of the PARABOLOID that the locus of the field of view is a sphere touching the paraboloid at the vertex, and whose radius is one-fourth the radius of curvature at that point; and, generally, whatever be the nature of the reflecting surface (provided it is symmetrical about a centre), the sphere here found will be the locus of the focus of rays incident near the centre of the mirror. I am indebted to Professor Cayley for the following proof of this interesting theorem:—

In ANY small mirror for parallel rays inclined to the axis in

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the principal plane, the locus of the focus is a circle whose diameter $=\frac{1}{2}$ radius of curvature at vertex. (See Figure *c.*)

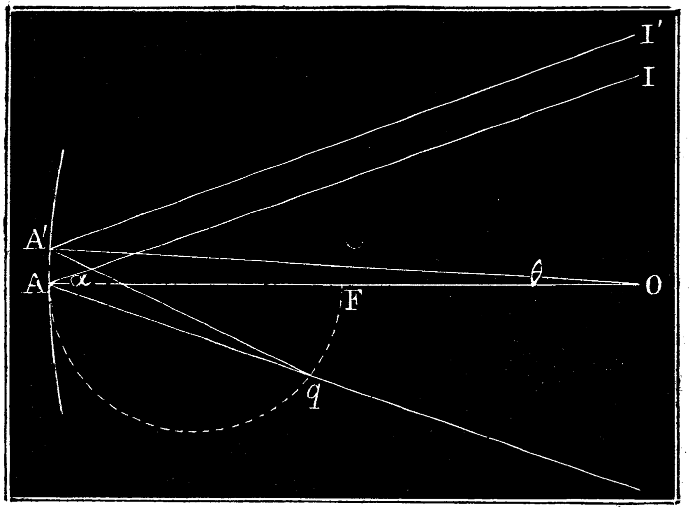

Fig. *c.*

a the radius of curvature O A.
α the inclination I A O.
θ the small angle A' O A.

The inclination of $A'q$ to axis $=\alpha+2\theta$; whence if $x\,y$ are coordinates of A', and X Y current coordinates,

Equation of

$$\begin{aligned} A'q \text{ is } Y-y &= -(X-x)\tan(\alpha+2\theta) \\ Aq \text{ is } Y &= -X\tan\alpha \\ x = a(1\ \ \cos\theta) &= -\tfrac{1}{2}a\theta^2 \\ y = a\sin\theta &= a\theta. \end{aligned}$$

Equations are

$$Y-y=-(X-x)(\tan\alpha+2\theta\,.\,\sec^2\alpha)$$

say

$$\begin{aligned} Y+X\tan\alpha &= a\theta-2X\sec^2\alpha\,.\,\theta \\ Y+X\tan\alpha &= 0 \end{aligned}$$

whence

$$a\theta-2X\sec^2\alpha\,.\,\theta=0,\ \text{i.e.}\ X=\tfrac{1}{2}a\cos^2\alpha\ \therefore Y'=-\tfrac{1}{2}a\sin\alpha\cos\alpha$$
$$X^2+Y^2=\tfrac{1}{4}a^2\cos^2\alpha=\tfrac{1}{2}aX,$$

which is circle on diameter A F (if A F$=\frac{1}{2}$ A O).

In the case of the paraboloidal mirror the unsymmetrical form of the visible images near the boundary of the field is attributable mainly to the curvature of the field-surface, which has of course the effect of throwing the non-central part of the field outside the focus of the eye-piece or outside the plane of the photographic plate, as the case may be. With a photographic plate ground to a *concave* surface of the proper curvature, there is reason to believe that symmetrical star images may be obtained in a field of the usual angular extent. But this is subject to the objection that the plate will always stop out the central and best part of the pencil.

4. In the SPHERICAL MIRROR the visible image will not coincide with the focus for rays incident at the centre of the mirror unless the angle subtended by the arc of the mirror is supposed to be small relatively to the angle of incidence of the oblique pencil. For the actual conditions of astronomical images the visible field will be the locus of the point F, or locus of the epicycloidal cusps, which are the points of greatest condensation for the respective pencils. For any given pencil the cusp of the caustic lies in the line where the ray is a radius of the spherical surface. Hence the geometric field is a spherical surface concentric with the mirror and having half the radius; and the visible field lies very near it.

5. Evidently a spherical mirror will not give symmetrical images, except near the centre, because it is only at the centre of the field that the visible images are formed by the central part of the mirror. Neither will the sphere give a flat field, though it will give a nearer approach to one than the parabola. Against this advantage we must set off the objection that, except at the centre of the field, its images are not formed by the central part of the mirror.

6. As to the CYCLOIDAL MIRROR, if the preceding analysis be correct, the locus of the cusps of the cycloidal caustic is a plane surface, but the images become unsymmetrical as they recede from the centre of the field. It is not meant to be asserted that the visible field will be a true plane, because the visible field is the locus of the circle of least aberration, which is not identical with the locus of the caustic cusps. But the field will be very nearly a plane surface.

It is right to add that the possibility of obtaining a flat field by reflexion has been doubted by mathematicians of high authority, and it must be left to the reader to form his own opinion of the validity of the reasoning by which the conclusion last announced has been reached.

On some Nebulæ hitherto suspected of Variability or Proper Motion.

By J. L. E. Dreyer, Ph.D.

The discrepancies met with in comparing observations of nebulæ by different observers are frequently so great that more or less positive assertions have naturally from time to time been made as to variability or changes in the objects. While many of these assertions can at once be dismissed as showing that too